# CDU High School Maths Competition

**Objectives:** This is two-day workshop at Charles Darwin University (CDU) for Year 10 students in the Northern Territory to demonstrate their insights and proficiency in solving mathematical problems. The students will also work in groups and present their findings together.

**Target**: Year 10 students in the Northern Territory

**Contacts:** Dr David Ompong (david.ompong@cdu.edu.au) and Dr Eva Ignatious (eva.ignatious@cdu.edu.au).

**Materials required for schools/students**: Flyers to be sent to schools and sample questions and solutions for students.

**Registration:**

1) Students will complete an individual registration. Students who can successfully attempt at least one sample question on the CDU webpage are encouraged to enter the competition.

2) Local Schools will register three students to represent their school for the group challenge.

**Date for the competition: TBC**

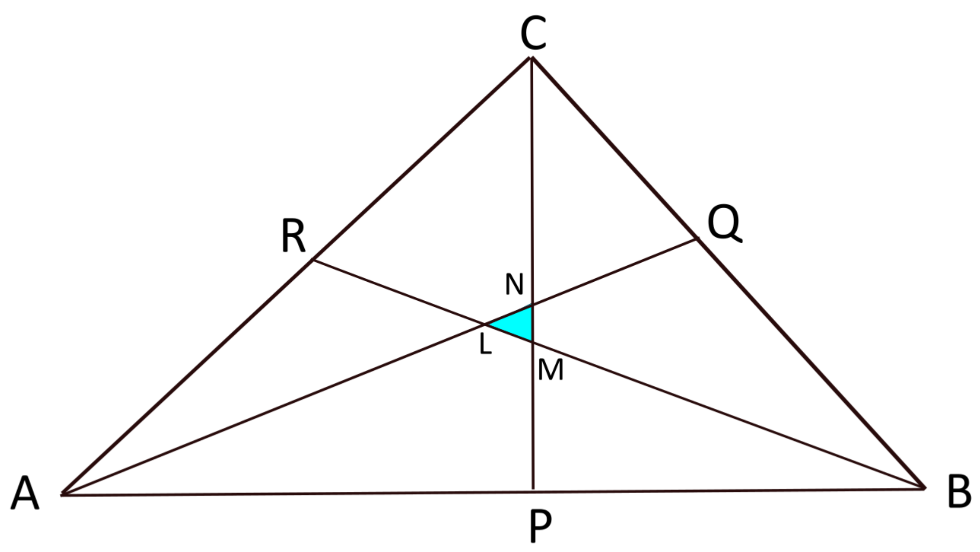
**Sample Questions**

1. Prove that: $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n + 1)^2$
2. Find the possible solutions of the given equation

$$x^3 - (3 + \sqrt{3})x = 0$$

3. Let *m* and *n* be 2 integers each one not divisible by 3. Use proof by cases to show that *mn* is not divisible by 3.

Given AB = 16 cm, BC = 10 cm & AC = 14 cm. RA = RC, AB ⊥ PC, AQ is angle bisector of ∠BAC calculate the area of the shaded triangle LMN. (Take $\tan \cos^{-1}(5/\sqrt{28}) = \frac{1}{5}\sqrt{3}$); $\sin [90 - \cos^{-1}((7\sqrt{129})/86)] = 7\sqrt{129}/86$)



**Suggested solutions**

1. **Step1** Show it is true for **n=1**

$1^3 = ¼ \times 1^2 \times 2^2$ is True

**Step 2** Assume it is true for **n=k**

$1^3 + 2^3 + 3^3 + ... + k^3 = ¼k^2(k + 1)^2$ is True (An assumption!)

**Step3** Now, prove it is true for "k+1"

$1^3 + 2^3 + 3^3 + ... + (k + 1)^3 = ¼(k + 1)^2(k + 2)^2$ ?

We know that $1^3 + 2^3 + 3^3 + ... + k^3 = ¼k^2(k + 1)^2$ (the assumption above), so we can do a replacement for all but the last term:

$¼k^2(k + 1)^2 + (k + 1)^3 = ¼(k + 1)^2(k + 2)^2$

Multiply all terms by 4:

$k^2(k + 1)^2 + 4(k + 1)^3 = (k + 1)^2(k + 2)^2$

All terms have a common factor $(k + 1)^2$, so it can be cancelled:

$k^2 + 4(k + 1) = (k + 2)^2$

And simplify:

$k^2 + 4k + 4 = k^2 + 4k + 4$

They are the same! So it is true.

So: $1^3 + 2^3 + 3^3 + ... + (k + 1)^3 = ¼(k + 1)^2(k + 2)^2$ is True

2. $x^3 - \left(3 + \sqrt{3}\right)x = 0$

Let y = $\sqrt{3}$, $y^2 = 3$

$x^3 - (y^2 + \mathrm{y}\ )x + \ y^2 = 0$

$x^3 - (y^2x + \mathrm{yx}\ )x + \ y^2 = 0$

$x^3 - y^2x + \mathrm{yx} + y^2 = 0$

$x(x^2 - y^2) - y(x - y) = 0$

$x(x - y)(x + y) - y(x - y) = 0$

$(x - y)(x(x + y) - y) = 0$

$(x - y)(x^2 + xy - y) = 0$

$(x - \sqrt{3})(x^2 + x\sqrt{3} - \sqrt{3}) = 0$

$x - \sqrt{3} = 0$ , $x = \sqrt{3}$

$(x^2 + x\sqrt{3} - \sqrt{3}) = 0$

$using$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-\sqrt{3} \pm \sqrt{\left(\sqrt{3}\right)^2 - 4(-\sqrt{3})}}{2}$$

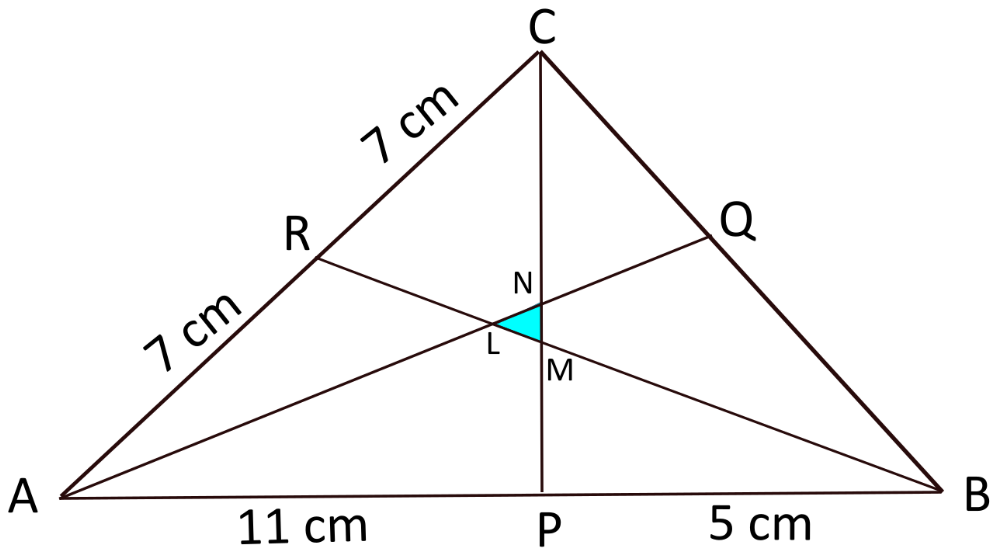
$$x = -\left(\frac{-\sqrt{3} \pm \sqrt{\left(\sqrt{3}\right)^2 - 4(-\sqrt{3})}}{2}\right)$$

**3.** There are 3 cases; consider each case in turn.

If $m = 3a + 1$ and $n = 3b + 1$, then $mn = (3a + 1)(3b + 1) = 9ab + 3a + 3b + 1$, which is not divisible by 3.

If $m = 3a + 1$ and $n = 3b + 2$, then $mn = (3a + 1)(3b + 2) = 9ab + 6a + 3b + 2$, which is not divisible by 3.

If $m = 3a + 2$ and $n = 3b + 2$, then $(3a + 2)(3b + 2) = 9ab + 6a + 6b + 4$, which is not divisible by 3.




**4.** From $\Delta ABC$
$AC^2 = AB^2 + BC^2 - 2\times AB\times BC\times\cos(\angle ABC)$ {Law of Cosines}
$\Rightarrow 14^2 = 16^2 + 10^2 - 2\times16\times10\cos(\angle ABC)$
$\Rightarrow 196 = 256 + 100 - 320\cos(\angle ABC)$
$\Rightarrow 320\cos(\angle ABC) = 160$
$\Rightarrow \cos(\angle ABC) = ½$
$\Rightarrow$ **$\angle ABC = 60°$**

From $\Delta BPC$
$\sin(\angle ABC) = PC/BC = PC/10$
$\Rightarrow \sin 60 = PC/10$
$\Rightarrow ½\sqrt{3} = PC/10$
$\Rightarrow$ **$PC = 5\sqrt{3}$ cm**

$\cos(\angle ABC) = PB/BC$
$\Rightarrow \cos 60 = PB/10$
$\Rightarrow ½ = PB/10$
$\Rightarrow$ **$PB = 5$ cm**
From $\Delta APC$
$AP = AB - PB$
$\Rightarrow AP = 16 - 5$
$\Rightarrow$ **$AP = 11$ cm**

$\sin(\angle PAC) = = PC/AC$
$\Rightarrow \sin(\angle PAC) = 5\sqrt{3}/11$

cos (∠BAC) = =AP/AC,

⇒ cos (∠BAC) = 11/14

Let ∠QAC = ∠QAP = x {AQ is angle bisector of ∠PAC}
cos (∠BAC) = cos 2x
⇒ cos (∠BAC) = 2 $\cos^2$ x – 1 (double angle identity for cosine)
⇒ 2 $\cos^2$ x - 1 = 11/14
⇒ $\cos^2$ x = 25/28
⇒ cos x = ± 5/√28
0 < x < 180 {x is an angle inside the triangle}
⇒ cos x = 5/√28

**⇒ ∠QAP = $\cos^{-1}$ (5/√28)**

From ΔARB
$BR^2 = AB^2 + AR^2$ - 2×AB×AR×cos (∠BAR)
⇒ $BR^2 = 16^2 + 7^2$ - 2×16×7×11/14
= 256 + 49 - 176 = 129

**⇒ BR = √129 cm**
$AR^2 = AB^2 + BR^2$ - 2×AB×BR×cos (∠ABR)
⇒ $7^2 = 16^2$ + 129 - 2×16×√129×cos (∠ABR)
⇒ 49 = 256 + 129 - 32√129 cos (∠ABR)
⇒ cos (∠ABR) = (7√129)/86

**⇒ ∠ABR = $\cos^{-1}$ ((7√129)/86)**

From ΔAPN
tan ∠QAC = tan $\cos^{-1}$ (5/√28)
⇒ PN/PA = tan $\cos^{-1}$ (5/√28)
⇒ PN/11 = tan $\cos^{-1}$ (5/√28)
= ⅕√3

**⇒ PN = ⅕×11√3 cm**
∠ANP = 90 -∠QAP
⇒ ∠ANP = 90 - $\cos^{-1}$ (5/√28)
⇒ ∠LNM = 90 - $\cos^{-1}$ (5/√28)
⇒ sin ∠LNM = sin [90 - $\cos^{-1}$ (5/√28)]
= cos $\cos^{-1}$ (5/√28)

**⇒ sin ∠LNM = 5/√28**

From ΔPBM
tan ∠PMB = PM/PB
⇒ PM/5 = tan ($\cos^{-1}$ ((7√129)/86))

= $5\sqrt{3}/21$

⇒ **PM = $25\sqrt{3}/21$ cm**

∠PMB = 90 -∠ABR

⇒ ∠PMB = 90 - $\cos^{-1}$ ((7√129)/86)

⇒ ∠LMN = 90 - $\cos^{-1}$ ((7√129)/86)

⇒ sin ∠LMN = sin [90 - $\cos^{-1}$ ((7√129)/86)]

= cos [$\cos^{-1}$ ((7√129)/86)]

⇒ **sin ∠LMN = 7√129)/86**

MN = PN - PM

⇒ MN = ⅕×11√3 - 25√3/21

⇒ **MN = 106√3/105 cm**

From ΔLMN

∠MLN = 180 - (∠LMN + ∠LNM)

⇒ ∠MLN = 180 - [90 - $\cos^{-1}$ ((7√129)/86) + 90 - $\cos^{-1}$ (5/√28)]

⇒ ∠MLN = $\cos^{-1}$ ((7√129)/86) + $\cos^{-1}$ (5/√28)

⇒ sin ∠MLN = sin [$\cos^{-1}$ ((7√129)/86) + $\cos^{-1}$ (5/√28)]

= sin [$\cos^{-1}$ ((7√129)/86)] cos $\cos^{-1}$ (5/√28) **(note: cos $\cos^{-1}$(x)= x)**

+ cos [$\cos^{-1}$ ((7√129)/86)] sin $\cos^{-1}$ (5/√28) **(note sin($\cos^{-1}$($x$))= $\sqrt{1-x^2}$ )**

= [(5√43)/86][5/√28] + [(7√129)/86][√3/√28]

⇒ **sin ∠MLN = 23√301/602**

LN/sin ∠LMN = MN/sin ∠MLN {Law of Sines}

⇒ LN = MN × sin ∠LMN/sin ∠MLN

⇒ LN = (106√3/105) × [7√129)/86]/[23√301/602]

⇒ **LN = 106√7/115 cm**

Area of ΔLMN = ½ × LN × MN × sin ∠LNM

Area of ΔLMN = ½ × (106√7/115) × (106√3/105) × (5/√28)

**Area of ΔLMN = 2809√3/2415 cm²**